Critical behavior of efficiency dynamics in small-world networks

Sheng-You Huang, Xian-Wu Zou,* Zhi-Jie Tan, Zhi-Gang Shao, and Zhun-Zhi Jin
Department of Physics, Wuhan University, Wuhan 430072, People's Republic of China
(Received 8 March 2003; published 11 July 2003)

Some dynamical processes in a small-world network shows a critical transition at a finite disorder ϕ_c of the network, in contrast with the geometrical properties that exhibit the critical behavior at $\phi_c = 0$. Although it has been pointed out in previous works that the transition is related to the structural properties of the network, it is still not very clear why the transition occurs at $\phi_c \neq 0$. In this paper we present a simple social model of efficiency dynamics in small-world networks, which also shows a transition at $\phi_c > 0$. We obtain the critical point with $\phi_c \approx 0.098$ from the finite-size analysis. It is found that both the geometrical properties of the network and the specific dynamical characters of the model contribute to the critical transition. This work is useful for understanding this kind of transition occurring in many dynamical processes in small-world networks.

DOI: 10.1103/PhysRevE.68.016107 PACS number(s): 89.75.Hc, 05.40.−a, 87.23.Ge

I. INTRODUCTION

Many social, biological, and communication systems can be cast into the form of complex networks [1,2]. Various models have been developed in order to describe the structure and properties of these networks [1–6]. Among these models, the small-world network, which was introduced by Watts and Strogatz [4,5], has recently attracted a great deal of attention [7–12]. The small-world network is based on a locally connected regular lattice in which a fraction ϕ of the links between neighboring sites are randomly replaced by new random links, thus creating long-range "shortcuts." Parameter ϕ can be used to characterize the disorder degree of the network. The small-world model captures two specific features of real neural, social, and technological networks [4]. On one hand, it has a relatively large clustering coefficient, like regular lattices. On the other hand, it has a very small average shortest path through the network between any two nodes, like random graphs.

Small worlds may play an important role in the study of the influence of the network structure upon the dynamics of many social processes, such as disease spreading, formation of public opinion, distribution of wealth, etc. [13–18]. Some efforts have been directed to investigating the structures and properties of small-world networks, such as scaling, percolation, etc. [19–23]. It has been shown that the geometrical properties, as well as certain statistical-mechanics properties, show a first-order transition at disorder $\phi = 0$ in the limit of large systems, $N \to \infty$ [24–26]. There exists a typical length $N^*(\phi) \sim \phi^{-1/d}$ in the small-world network, where d is the dimension of the basic regular lattice. For the system with size above N^*, the network is indeed a small world and below N^* it behaves as a regular lattice [26]. That is, any finite value of the disorder induces the small-world behavior.

Recently, several works [27–29] have shown that some dynamical processes in the small-world network exhibit the critical transition at a finite value of ϕ, which is distinguished from the transition occurring at $\phi = 0$. Kuperman and Abramson have studied an epidemiological model in a small-world network [27]. It shows that there exists a transition from a fluctuating epidemic state of low infection to a self-sustained oscillation one at a finite value of ϕ. Zanette has studied the critical behavior of rumor propagation in a small-world network [28]. The transition occurs between a regime where the rumor "dies" in a small neighborhood of its origin and a regime where it spreads over a finite fraction of the whole population. In a very recent paper by our group, we have also found that there exists a transition from a sparse-inactive state to a dense-active one of "life" at a finite disorder ϕ in a small-world network [29]. All these dynamical processes in a small-world network all show a critical transition at $\phi_c \neq 0$. Why does the transition take place at an intermediate ϕ value but not at $\phi = 0$? This is still not very clear up to date. In the previous works, an explanation about this transition involves only the geometrical properties of the network, such as the clustering coefficient [27–29]. In a recent work [28], it is suggested that in addition to the geometrical properties of the network, the specific dynamical characters of the studied model must be taken into account to explain the origin of the transition.

In this paper we use a simple model that describes the dynamics of efficiencies of competing agents [30] in a small-world network. Agents communicate, leading to the increase of efficiencies of underachievers, and the efficiency of each agent can increase or decrease irrespective of other agents. The model can also be considered as a polynuclear growth model with desorption, where the degrees of freedom are the heights of a growing interface [31–33]. The model shows a delocalization transition from a stagnant phase to a growing one at a finite disorder of the network. By taking into account the specific dynamical properties of the model, we predict the appearance of this critical transition. The present work will be useful for understanding the critical transition appearing in many dynamical processes at a certain finite disorder in a small-world network.

II. MODEL AND METHOD

In the present model, the evolution of the efficiencies is similar to that used in Ref. [30], which may mimic the dy-

*Author to whom correspondence should be addressed. Email address: xwzou@whu.edu.cn

namics of efficiencies of competing agents such as airlines, travel agencies, insurance companies, etc. The efficiency of each agent is expressed as a single non-negative number. The efficiency of every agent can, independent of other agents, increase or decrease stochastically by a certain amount which we set equal to unity. The interactions between the agents of the population are described by a small-world network. The vertices denote the agents and the links represent the business relationships between subjects. The efficiency of an agent can only be affected by its linked agents. As in the Watts and Strogatz model [4], the small worlds used in this work are random networks built upon a topological ring with N vertices and coordination number $2K$. Each link, connecting a vertex to its neighbor in the clockwise sense, is then rewired at random, with probability ϕ, to any vertex of the systems. With probability $(1-\phi)$, the original link is preserved. Self-connections and multiple connections are prohibited. With this procedure, we obtain a regular lattice at $\phi=0$, and progressively random graphs for $\phi>0$. The long-range links that appear at any $\phi>0$ trigger the small-world phenomenon. At $\phi=1$ all the links have been rewired and the system is similar to a completely random network. To avoid producing disconnected graphs, we have chosen $K=2$ in the present model.

Now we define the efficiency model as follows. Each vertex i in the network represents an agent which is characterized by a non-negative integer $h_i(t)$. This integer stands for their efficiency level. That is, the higher h_i is, the more advanced (efficiently speaking) the agent is. We assume that the interaction equates the efficiencies of underachievers to the efficiencies of better performing agents. Similar to Ref. [30], the calculated results are expected to be independent of the initial conditions for the present model. For simplicity, we choose the efficiency of each agent $h_i(0)=0$ as the initial conditions. Monte Carlo (MC) simulations have been used to study the evolution of the efficiencies of N agents in the small world network. At each MC step, agent i is selected at random and updates the agent's efficiency level as follows:

(i) $h_i(t)\rightarrow\max[h_i(t),h_j(t)]$ with probability $1/(1+p+q)$, where agent j is one of the agents that are linked to agent i. This move is due to the fact that each agent tries to equal his efficiency to that of a better performing agent in order to stay competitive.

(ii) $h_i(t)\rightarrow h_i(t)+1$ with probability $p/(1+p+q)$. This incorporates the fact that each agent can increase his efficiency, say due to innovations, irrespective of other agents.

(iii) $h_i(t)\rightarrow h_i(t)-1$ with probability $q/(1+p+q)$. This corresponds to the fact that each agent can lose his efficiency due to unforeseen problems such as labor strikes. Note, however, that since $h_i(t)\geqslant 0$, this move can occur only when $h_i(t)\geqslant 1$.

Then, the evolution of efficiency comes into the next MC step. After each MC step the "time" is increased by $1/N$, such that after one time step, on the average, all agents in the network have made an update. In order to investigate the effect of the topological structure of the population on the dynamics of efficiencies, we fix the value of parameters p and q in the present model. Without loss of generality, we choose $p=3/2$ and $q=15/2$, where the mean-field theory predicts a growing phase of efficiencies [30].

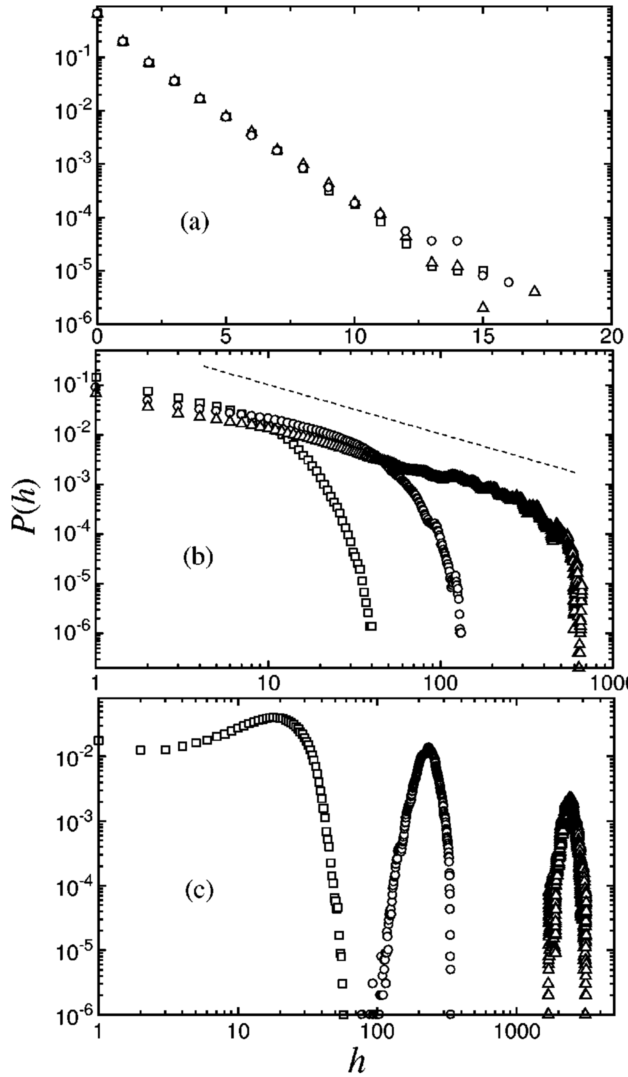

FIG. 1. Efficiency distribution $P(h)$ for coordination number $2K=4$ and disorder $\phi=0.05$ (a), 0.175 (b), and 0.30 (c). Different symbols correspond to times $t=10^3$ (square), 10^4 (circle), and 10^5 (triangle). The dashed line in (b) has a slope of -1.0. Note carefully the different scales of the three plots. The size of the system is $N=1000$.

III. RESULTS AND DISCUSSION

We have performed extensive numerical simulations to investigate the dynamics of efficiency in the small-world networks with size N ranging from 200 to 1 000 000 and rewiring probabilities $\phi\in[0,1]$. To reduce the effect of fluctuation on calculated results, for every system with size N, the calculated results are averaged over both n different network realizations and 10 independent runs for each network realization, in such a way that $n\times N\approx 1\times 10^5$.

We have studied the distribution of the agents with efficiency h at a set of times t. Figure 1 shows the normalized distribution $P(h,t)$ as a function of efficiency h for the sys-

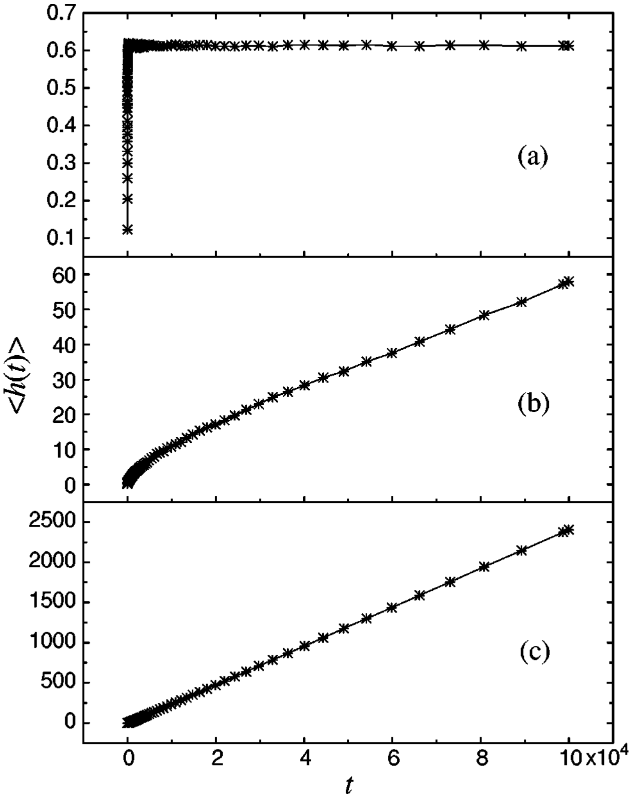

FIG. 2. Average efficiency $\langle h(t)\rangle$ as a function of time t for disorders $\phi=0.05$ (a), 0.175 (b), and 0.30 (c). The size of the system is $N=1000$.

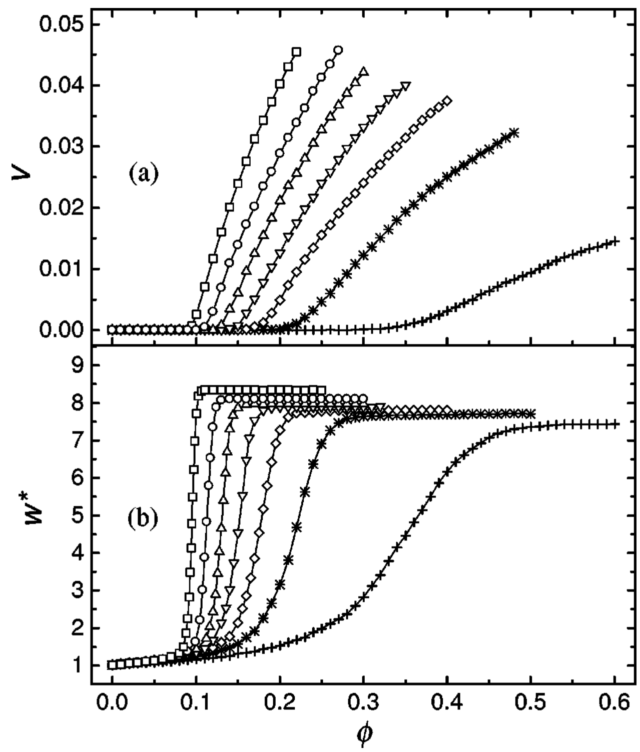

FIG. 3. (a) The asymptotic growth rate v of the average efficiency and (b) the asymptotic efficiency fluctuation w^* as a function of disorder ϕ of the network. From left to right, the system size is $N=1\,000\,000$, 20 000, 5000, 2000, 1000, 500, and 200.

tem with size $N=1000$ at three different values of ϕ. It can be seen from Fig. 1 that as disorder ϕ increases, distribution $P(h,t)$ has a large change. As the disorder is small (e.g., $\phi=0.05$), the distribution, approximately, has an exponential form and is independent of time t [see Fig. 1(a)]. In this regime, because of the lack of the long-range link, the increase of efficiency due to the communications among agents is small and it is comparable to the reduction of efficiency. The distribution approximately reaches a time-independent steady state, and thereby, the average efficiency per agent approaches a low constant in the long time [see Fig. 2(a)]. This kind of steady exponential distribution has been predicted by the mean-field theory in Ref. [33]. However, as the disorder is large (e.g., $\phi=0.30$), the distribution has a Gaussian form and the position of its maximum increases with time t [see Fig. 1(c)]. In this situation, the existence of a large number of long-range links makes the cooperation and interchange very easy and effective, so each agent can reach the efficiency of the better agents. Thus, the distribution has a Gaussian form and the efficiency of agents will linearly increase with time t [see Fig. 2(c)], which is consistent with the mean-field analysis in Ref. [30]. In the case of intermediate disorder (e.g., $\phi=0.175$), just before the Gaussian peak begins to appear, the distribution of efficiency follows a power-law dependence, i.e., $P(h)\sim h^{-\alpha}$ with $\alpha\approx1.0$, over a certain region of efficiencies for different times, as shown in Fig. 1(b).

The appearance of a well-defined power-law distribution indicates that there may exist a critical phase transition from the regime where the efficiency distribution is stationary, to the regime where the efficiency increases with time at a certain intermediate value of ϕ. To characterize this transition, we calculate growth rate v of average efficiency $\langle h(t)\rangle$ per agent in the long-time limit, where

$$v\equiv\frac{d\langle h(t)\rangle}{dt} \tag{1}$$

and

$$\langle h(t)\rangle=\frac{1}{N}\sum_{i=1}^{N}h_i(t)=\sum_{h=0}^{\infty}hP(h,t). \tag{2}$$

As the disorder is small, efficiency distribution $P(h,t)$ is independent of time t and can be rewritten as $P(h)$. Thus, we have $\langle h(t)\rangle\rightarrow$const. and $v\rightarrow0$ in the long-time limit. For large ϕ, the Gaussian-type efficiency distribution shifts to large efficiency with the increase of time t. Therefore, average efficiency $\langle h(t)\rangle$ is a function of time t and the corresponding growth rate $v>0$.

Figure 3(a) shows growth rate v of the average efficiency as a function of disorder ϕ for several systems with size N ranging from 200 to 10^6. It can be seen from this figure that there exists a transition at a certain value $\phi_c(N)$ for each system. As $\phi<\phi_c(N)$, growth rate v is equal to zero; as $\phi>\phi_c(N)$, v increases rapidly with ϕ. As $\phi\approx\phi_c(N)$, growth rate v transits from zero to a finite value, which corresponds to the transition of the system from a stagnant phase to a growing one. This transition can be also characterized by efficiency fluctuation w of the system, which corresponds to the nonuniform degree of efficiencies in the system. Efficiency fluctuation $w(t)$ is expressed as

$$w^2(t)=\frac{1}{N}\sum_{i=1}^{N}[h_i(t)-\langle h(t)\rangle]^2. \tag{3}$$

Efficiency fluctuation $w(t)$ tends to a constant $w^*=\langle w(t\rightarrow\infty)\rangle$ in the long-time limit. Figure 3(b) shows the asymptotic value w^* as a function of disorder ϕ for several systems with size N ranging from 200 to 10^6. From Fig. 3(b) we can see that fluctuation w^* also shows a transition behavior similar to that of growth rate v. As $\phi<\phi_c(N)$, fluctuation w^* takes a small value of about 1.0; as $\phi>\phi_c(N)$, fluctuation w^* becomes the maximum, close to 8.0; as $\phi\approx\phi_c(N)$, fluctuation w^* sharply jumps from a small value to the maximum one. The results in Fig. 3 confirm that the present efficiency model exhibits a delocalization transition from a stationary phase to a growing one of efficiencies at a certain intermediate ϕ value.

Figure 3 also shows that critical points $\phi_c(N)$ obtained from the finite-size systems are dependent on size N of systems. The apparent critical point $\phi_c(N)$ in a finite-size system shows a deviation from the true critical value $\phi_c(\infty)$, which corresponds to the critical point for the system with size $N\rightarrow\infty$. Smaller the system size N is, smoother the transition of the curve is, and larger the deviation $\phi_c(N)-\phi_c(\infty)$. To obtain the true critical point $\phi_c(\infty)$, we employ the finite-size analysis for the obtained data. From Fig. 3(b), we can estimate critical values $\phi_c(N)$ for the systems with different sizes corresponding to the inflexions of the curves. The results of $\phi_c(N)$ are shown in Fig. 4(a) on a log-log plot. It can be seen from Fig. 4(a) that with the increase of system size N, critical value $\phi_c(N)$ decreases and tends to a constant value that corresponds to the true critical value $\phi_c(\infty)$ for the infinite-size system. According to the finite-size effects of the systems, the apparent critical point $\phi_c(N)$ and true critical point $\phi_c(\infty)$ are expected to scale with size N as [34]

$$\phi_c(N)-\phi_c(\infty)\sim N^{-1/\nu}, \tag{4}$$

where ν is the critical shift exponent. To obtain the value of true critical point $\phi_c(\infty)$ and critical exponent ν, Fig. 4(b) shows critical deviation $\phi_c(N)-\phi_c(\infty)$ as a function of system size N on a log-log plot. When the true critical value is chosen to be $\phi_c(\infty)\simeq0.098$, we obtain the best power-law relation of the data by using Eq. (4) [see Fig. 4(b)]. The excellent linear dependence in Fig. 4(b) indicates that the finite-size scaling relation Eq. (4) is reasonable for describing the present simulation results. From Fig. 4(b) we also obtain critical exponent $\nu\simeq1.75$ by means of the least-square fit to the data.

The obtained data show that there exists a phase transition in the model of dynamics of efficiency at a finite disorder ϕ_c of the network. The finite-size scaling analysis supports the presence of the critical phenomenon at finite ϕ_c. This kind of critical behavior is also found in other systems such as

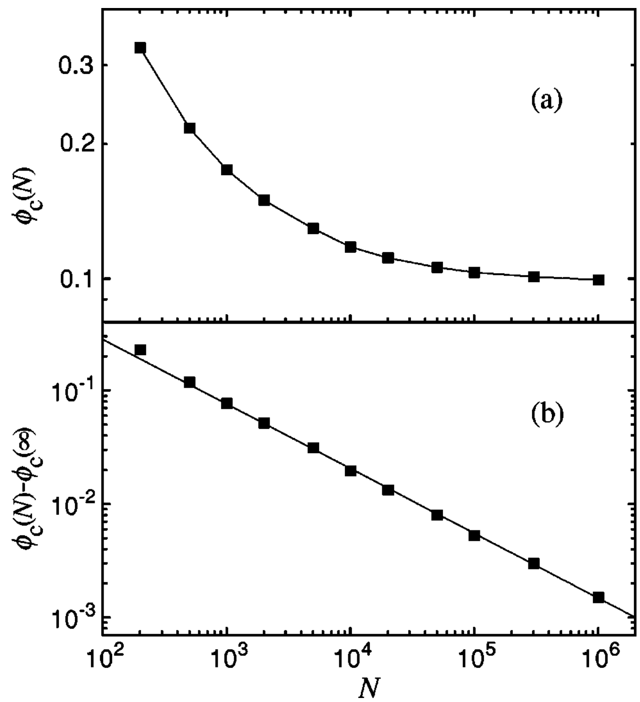

FIG. 4. (a) Critical disorder $\phi_c(N)$ for finite-size systems as a function of system size N on a log-log plot. (b) Deviation $\phi_c(N)-\phi_c(\infty)$ from the true critical value as a function of size N on a log-log plot, where $\phi_c(\infty)$ is chosen to be 0.098. The symbols are the simulation results, and the line is the least-square fit to the data.

epidemic dynamics, rumor propagation, and the game of Life in the small-world network [27–29]. It is well known that the geometrical properties, such as the average shortest path $\overline{\ell}(\phi)$, show a first-order transition at disorder $\phi=0$ [24–26]. Therefore, the present transition occurring at $\phi_c\neq0$ cannot be attributed to the purely geometrical properties of the network, and the specific dynamical characters of the model should be taken into account. In the following, we try to understand the present critical behavior from the dynamical properties of the model.

First, we write down the evolution equation for average efficiency $\langle h(t)\rangle$ per agent. In the present model, the contributions to the time evolution of $\langle h(t)\rangle$ come from three parts: increase due to learning from its linked agents, increase due to innovation, and decrease due to unforeseen problems. Thus growth rate v of the average efficiency can be expressed as [30]

$$v(t)\equiv\frac{d\langle h(t)\rangle}{dt}=\frac{Aw(t)+p-qs(t)}{1+p+q}, \tag{5}$$

where A is a proportional factor concerned with disorder ϕ, and $s(t)$ is the probability that an agent has a nonzero efficiency. The first term on the right-hand side of the above equation indicates the increase in efficiency per agent due to the fact that each agent tries to equal its efficiency to that of a better performing agent, which is proportional to the nonuniform degree w^* of efficiencies among agents. The second term represents the increase in efficiency per agent due to the innovation of each agent. The last term quantifies the loss in efficiency per agent due to some unforeseen problems, taking into account the fact that the reduction can take place from an agent only if the agent has a nonzero efficiency. Substituting the values of p and q, Eq. (5) becomes

$$v(t)=\frac{1}{20}[2Aw(t)+3-15s(t)]. \tag{6}$$

From Eq. (6) we can see that there should be a critical transition at an intermediate disorder ϕ_c of the network. As $\phi<\phi_c$, factor A is small because of finite communication among agents. The efficiencies of all the agents are not high, and the corresponding fluctuation $w(t)$ is also small. In this regime, one can expect that in the long-time limit $t\rightarrow\infty$, the first two terms and the last term on the right-hand side of the above equation will cancel each other and the probability with nonzero efficiency reaches the asymptotic time-independent value, i.e., $s=(2Aw^*+3)/15$. This indicates that growth rate $v=0$ and average efficiency per agent $\langle h\rangle$ becomes a constant in the long-time limit. Correspondingly, the steady-state efficiency distribution has an exponential form of $P(h)\sim\exp(-h/h^*)$ with a finite first moment $\langle h\rangle$ [33]. We call this phase the ‘‘stagnant’’ phase. However, as $\phi>\phi_c$, the proportional factor A and w are large due to the abundant long-range links; after a long time, w attains the stable value w^* and the probability with nonzero efficiency reaches the maximum value of $s=1$, but the last term on the right-hand side of Eq. (6) is still less than the sum of the first two terms. In this regime, growth rate $v=[2Aw^*-12)]/15$ and average efficiency per agent $\langle h(t)\rangle$ increases linearly with time t, i.e., $\langle h(t)\rangle=[2Aw^*-12)t/15$. Correspondingly, the efficiency distribution has a Gaussian form of $P(h)\sim\exp[-(h-h_0)^2/\Delta^2]$ with a finite width Δ [30]. We call this phase the ‘‘growing’’ phase. In this phase, the correlations between h_i's of different agents approximate to zero and the efficiency of each agent increases independently [30], This may lead to the ϕ-independent efficiency fluctuation w^*.

IV. CONCLUSIONS

We investigate a simple model of the dynamics of efficiencies of competing agents in a small-world network. The results show that there exists a delocalization (or depinning) phase transition from a stagnant phase to a growing one at a finite disorder ϕ_c of the network. Above the critical point $\phi>\phi_c$, the average efficiency increases linearly with time; below it ($\phi\leqslant\phi_c$) the system is stagnant, i.e., the efficiency distribution becomes stationary in the long-time limit and the average efficiency per agent approaches a constant. By means of a finite-size scaling analysis, we obtain critical point $\phi_c\simeq0.098$ for the given system. The present transition occurring at a finite disorder is different from the transition related to the geometrical properties of the network, which takes place at $\phi=0$. We predict this transition occurring at a finite disorder by counting in both the geometrical properties of the network and the specific dynamical properties of the model.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China Grant No. 10274056 and Fund of the Education Ministry of China.

[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S.N. Dorogovtsev and J.F.F. Mendes, Adv. Phys. **51**, 1079 (2002).
[3] A.-L. Barabasi and R. Albert, Science **286**, 509 (1999).
[4] D.J. Watts and S.H. Strogatz, Nature (London) **393**, 440 (1998).
[5] D.J. Watts, *Small Worlds* (Princeton University Press, Princeton, 1999).
[6] M.E.J. Newman, C. Moore, and D.J. Watts, Phys. Rev. Lett. **84**, 3201 (2000).
[7] M. Barahona and L.M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[8] E. Almaas, R.V. Kulkarni, and D. Stroud, Phys. Rev. Lett. **88**, 098101 (2002).
[9] A.D. Sanchez, J.M. Lopez, and M.A. Rodriguez, Phys. Rev. Lett. **88**, 048701 (2002).
[10] V. Latora and M. Marchiori, Phys. Rev. Lett. **87**, 198701 (2001).
[11] L.F. Lago-Fernandez, R. Huerta, F. Corbacho, and J.A. Siguenza, Phys. Rev. Lett. **84**, 2758 (2000).
[12] J. Davidsen, H. Ebel, and S. Bornholdt, Phys. Rev. Lett. **88**, 128701 (2002).
[13] F. Jasch and A. Blumen, Phys. Rev. E **63**, 041108 (2001); **64**, 066104 (2001).
[14] F. Jasch and A. Blumen, J. Chem. Phys. **117**, 2474 (2002).
[15] T. Koslowski, M. Koblischke, and A. Blumen, Phys. Rev. B **66**, 064205 (2002).
[16] R. Pastor-Satorras and A. Vespignani, Phys. Rev. E **63**, 066117 (2001).
[17] N. Zekri and J.P. Clerc, Phys. Rev. E **64**, 056115 (2001).
[18] M. Kuperman and D.H. Zanette, Eur. Phys. J. B **26**, 387 (2002).
[19] M.E.J. Newman and D.J. Watts, Phys. Rev. E **60**, 7332 (1999).
[20] M.E.J. Newman, I. Jensen, and R.M. Ziff, Phys. Rev. E **65**, 021904 (2002).
[21] C. Moore and M.E.J. Newman, Phys. Rev. E **61**, 5678 (2000).
[22] R.V. Kulkarni, E. Almaas, and D. Stroud, Phys. Rev. E **61**, 4268 (2000).
[23] C.F. Moukarzel, Phys. Rev. E **60**, R6263 (1999).
[24] A. Barrat and M. Weigt, Eur. Phys. J. B **13**, 547 (2000).
[25] M. Argollo de Menezes, C.F. Moukarzel, and T.J.P. Penna, Europhys. Lett. **50**, 574 (2000).
[26] M. Barthelemy and L.A.N. Amaral, Phys. Rev. Lett. **82**, 3180 (1999).
[27] M. Kuperman and G. Abramson, Phys. Rev. Lett. **86**, 2909 (2001).
[28] D.H. Zanette, Phys. Rev. E **65**, 041908 (2002); **64**, 050901(R) (2001).

[29] S.-Y. Huang, X.-W. Zou, Z.-J. Tan, and Z.-Z. Jin, Phys. Rev. E **67**, 026107 (2003).
[30] S.N. Majumdar and P.L. Krapivsky, Phys. Rev. E **63**, 045101(R) (2001).
[31] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).
[32] H. Hinrichsen, R. Livi, D. Mukamel, and A. Politi, Phys. Rev. Lett. **79**, 2710 (1997).
[33] S.N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. E **61**, 6337 (2000).
[34] M.N. Barber, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic Press, New York, 1983), Vol. 8.